Journal of Sound and Vibration (2000) **230**(3), 507–520
doi:10.1006/jsvi.1999.2626, available online at http://www.idealibrary.com on IDEAL®

GENERATION OF CUMULATIVE SUM FREQUENCY AND DIFFERENCE FREQUENCY ACOUSTIC WAVES IN A TWO-DIMENSIONAL HARD-WALLED WAVEGUIDE

M.-X. Deng

Physics Department, Logistics Engineering University, Chongqing 400016, *People's Republic of China*

(*Received* 12 *April* 1999, *and in final form* 28 *July* 1999)

An approach based on second-order approximation technique and partial wave analysis method has been proposed for analyzing the generation of the cumulative sum frequency (SF) and difference frequency (DF) acoustic waves in a two-dimensional hard-walled waveguide. There are two or more sets of fundamental modes with different frequencies, in the waveguide, if an excitation source is of multi-frequency. The cross-interaction between two partial waves of the fundamental modes with different frequencies causes the generation of the driven SF and DF waves. The driven SF and DF waves may have the cumulative growth effect once the two fundamental modes with different frequencies have the same phase velocities. With appropriate boundary and initial conditions of excitation, the physical process of generation of the cumulative SF and DF waves has been clearly shown, and the analytical solution of the cumulative SF and DF waves has also been determined. The solution shows that the cumulative SF and DF acoustic fields may be symmetrical or antisymmetrical. The numerical results clearly reveal the distortion and the symmetry of the field patterns of the cumulative SF and DF waves.

1. INTRODUCTION

The hard-walled waveguide filled with fluid is one of the simplest waveguides. Due to the bulk non-linearity of the fluid, second order acoustic wave generation will occur once a fundamental mode propagates in the waveguide. If the thickness of the waveguide is much less than its transverse width, for simplicity, the waveguide can be assumed to be two dimensional. Thus, the examination of second order acoustic wave generation in a two-dimensional hard-walled waveguide can be of practical significance. The generation of second order acoustic waves in a two-dimensional hard-walled waveguide has drawn more and more attention, and has been discussed intensively in the past [1–4]. So far, much attention has been drawn to the case in which the excitation source is of single frequency. In practical cases, however, the excitation source may include two or more frequencies, and two or more sets of acoustic propagation modes with different frequencies will be generated. Due to the bulk non-linearity of the fluid in the waveguide, second order

0022-460X/00/080507 + 14 $35.00/0

acoustic waves [including the second harmonic, the sum frequency (SF) and difference frequency (DF) acoustic waves] will occur, and, in general, it is difficult to obtain an analytical expression for the harmonic fields to predict and demonstrate their non-linear characteristics.

Moreover, previous analyses were focused on the cumulative second-harmonic generation caused by an excitation source with a single frequency. From those analysis results we still do not comprehend the cumulative growth effect of the SF and DF waves that cannot be avoided in practical case. Hence, the examination on the physical process of generation of the cumulative SF and DF waves is necessary.

Usually, analyses are focused on the cumulative growth of second order acoustic waves due to their obvious effect. This article, under a quadratic perturbation, derives a full solution for the generation of the cumulative SF and DF waves caused by an excitation source with two different frequencies. The solution should work for any excitation source with two different frequencies in contrast to the previous analysis in which the source is of single frequency. For simplicity, we still assume that the fluid in the waveguide is homogeneous with no attenuation, no dispersion and no mean flow, and that there is no energy exchange between the fundamental waves and the higher harmonics, and that second order wave amplitude is much smaller than the fundamental one which we suppose as constant. The analysis results reveal that the SF and DF waves will have the cumulative effect once two fundamental modes with different frequencies have the same phase velocities, and that the symmetrical characteristics of the cumulative SF and DF acoustic field patterns are dependent of those of two fundamental modes.

2. THEORETICAL FUNDAMENTAL

A Cartesian co-ordinates system is shown in Figure 1, in which the oz-axis coincides with the center of the hard-walled waveguide, and the oy-axis is normal to the walls of the waveguide. It is assumed that an excitation source at position $z = 0$ has two frequencies f_1 and f_2. Each frequency corresponds to a set of fundamental modes. For simplicity, the sign, (f_i, l), is used to indicate the lth fundamental mode with frequency f_i $(i = 1, 2)$. According to the partial wave analysis method [5], the lth fundamental mode with frequency f_i and angular frequency ω_i consists of two partial longitudinal waves which are reflected at the upper and lower walls of the waveguide. Because there is a condition of phase matching between the two partial longitudinal waves of the (f_i, l) mode, the oz-axis components of two wave vectors are of the same magnitude. The formal solution of two partial longitudinal waves of the (f_i, l) mode is given by [4, 5]

$$\mathbf{u}^{(1)}_{(f_i,l)-1} = u_{(f_i,l)-1}\mathbf{K}^0_{(f_i,l)-1}\exp[\mathrm{j}\mathbf{K}_{(f_i,l)-1}\cdot\mathbf{r}_1 - \mathrm{j}\omega_i t],$$

$$\mathbf{u}^{(1)}_{(f_i,l)-2} = u_{(f_i,l)-2}\mathbf{K}^0_{(f_i,l)-2}\exp[\mathrm{j}\mathbf{K}_{(f_i,l)-2}\cdot\mathbf{r}_2 - \mathrm{j}\omega_i t] \quad (1)$$

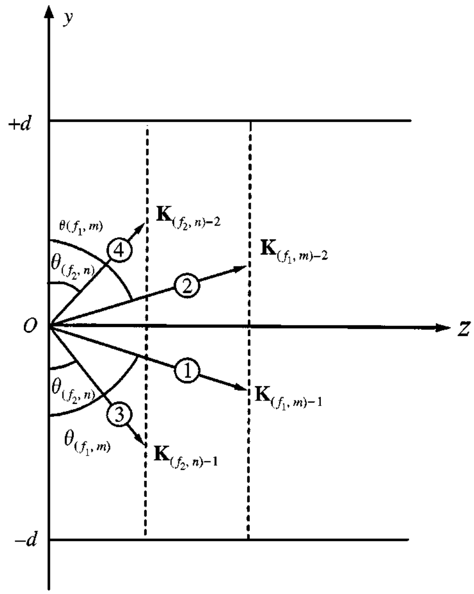

Figure 1. The wave vectors and the mechanical displacement ones of the (f_1, m) and (f_2, n) modes, ①-$\mathbf{u}^{(1)}_{(f_1,m)-1}$, ②-$\mathbf{u}^{(1)}_{(f_1,m)-2}$, ③-$\mathbf{u}^{(1)}_{(f_2,n)-1}$, and ④-$\mathbf{u}^{(1)}_{(f_2,n)-2}$.

with

$$K_{f_i} = |\mathbf{K}_{(f_i,l)-1}| = |\mathbf{K}_{(f_i,l)-2}| = \omega_i/c,$$

$$k_{(f_i,l)} = K_{f_i}\sin\theta_{(f_i,l)},\ \ \alpha_{(f_i,l)}k_{(f_i,l)} = K_{f_i}\cos\theta_{(f_i,l)},$$

$$c_{(f_i,l)} = \omega_i/k_{(f_i,l)} = c\sin\theta_{(f_i,l)},$$

$$\mathbf{K}_{(f_i,l)-p}\cdot\mathbf{r}_p = k_{(f_i,l)}z + (-1)^p\alpha_{(f_i,l)}k_{(f_i,l)}y, \quad p = 1, 2. \tag{2}$$

In equations (1) and (2), $\mathbf{u}^{(1)}_{(f_i,l)-p}$ $(p = 1, 2)$ is the mechanical displacement vector of the partial longitudinal wave, $u_{(f_i,l)-p}$ is the amplitude of $\mathbf{u}^{(1)}_{(f_i,l)-p}$, $\mathbf{K}^0_{(f_i,l)-p}$ is the unit vector of the wave vector $\mathbf{K}_{(f_i,l)-p}$, K_{f_i} is the magnitude of $\mathbf{K}_{(f_i,l)-p}$, c is the longitudinal velocity of the fluid, and $c_{(f_i,l)}$ denotes the phase velocity of the (f_i, l) mode.

The ultimate displacement vector of the (f_i, l) mode is given by

$$\mathbf{u}^{(1)}_{(f_i,l)} = \mathbf{u}^{(1)}_{(f_i,l)-1} + \mathbf{u}^{(1)}_{(f_i,l)-2}. \tag{3}$$

The boundary condition requires that the oy-axis component of $\mathbf{u}^{(1)}_{(f_i,l)}$ equal zero at $y = \pm d$, which yields the following matrix equation:

$$[M(k_{(f_i,l)}d)]\begin{bmatrix} u_{(f_i,l)-1} \\ u_{(f_i,l)-2} \end{bmatrix} = 0, \tag{4}$$

where $[M(k_{(f_i,l)}d)]$ is a 2×2 matrix the elements of which are given in Appendix A. From $|M(k_{(f_i,l)}d)| = 0$ we have the equations.

$$2\alpha_{(f_i,l)} k_{(f_i,l)} d = l\pi \quad (l = 0, 1, 2, \dots) \tag{5}$$

and

$$u_{(f_i,l)-1} = (-1)^l u_{(f_i,l)-2}. \tag{6}$$

The amplitude $u_{(f_i,l)-p}$ $(p = 1, 2)$ can be determined by the excitation source. The wave vectors and the mechanical displacement ones of the partial waves associated with the (f_1, m) and (f_2, n) modes are illustrated in Figure 1.

A second order wave equation of the fluid in the Cartesian co-ordinates system is given by [4, 6]

$$c^2 \nabla(\nabla \cdot \mathbf{u}^{(2)}) - \mathbf{u}^{(2)}_{,tt} = \mathbf{F}(\mathbf{u}^{(1)}),$$

$$\mathbf{F}(\mathbf{u}^{(1)}) = \tfrac{1}{2} c^2 (1 + \gamma) \nabla(\nabla \cdot \mathbf{u}^{(1)})^2, \tag{7}$$

where γ is the ratio of specific heats for the gas or $(1 + B/A)$ for the liquid, $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ denote the fundamental and second order harmonic displacements, respectively, $\mathbf{F}(\mathbf{u}^{(1)})$ is the driving force produced by the fundamental wave $\mathbf{u}^{(1)}$ due to the bulk non-linearity of the fluid.

In fact, only the real part of $\mathbf{u}^{(1)}_{(f_i,l)-p}$ $(p = 1, 2)$ has physical meaning in a practical problem. Thus, the ultimate displacement vector $\mathbf{u}^{(1)}$ of the (f_1, m) and (f_2, n) modes can be expressed as

$$\mathbf{u}^{(1)} = \frac{1}{2} \sum_{p=1,2} [\mathbf{u}^{(1)}_{(f_1,m)-p} + \tilde{\mathbf{u}}^{(1)}_{(f_1,m)-p} + \mathbf{u}^{(1)}_{(f_2,n)-p} + \tilde{\mathbf{u}}^{(1)}_{(f_2,n)-p}], \tag{8}$$

where $\tilde{\mathbf{u}}^{(1)}_{(f_i,l)-p}$ $(i, p = 1, 2, l = m, n)$ is the complex conjugation of $\mathbf{u}^{(1)}_{(f_i,l)-p}$. Inserting $\mathbf{u}^{(1)}$ into $\mathbf{F}(\mathbf{u}^{(1)})$ yields the multi-component of the driving force. Because the cumulative second-harmonic generation arising from the partial wave self-interaction of each fundamental mode has been intensively studied before [1–4], we only taken into account the partial wave cross-interaction between the (f_1, m) and (f_2, n) modes. Thus, the corresponding driving force component can be formally written as [neglecting the factor $\exp[-\mathrm{j}(\omega_1 \pm \omega_2)t]$]

$$\mathbf{F}(\mathbf{u}^{(1)}) = \sum_{p,q=1,2} F_{(f_1 \pm f_2,m,n)pq} \mathbf{K}^0_{(f_1 \pm f_2,m,n)pq} \exp[\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pq} \cdot \mathbf{r}_{pq}] + \text{c.c.} \tag{9}$$

with

$$\mathbf{K}_{(f_1 \pm f_2,m,n)pq} = \mathbf{K}_{(f_1,m)-p} \pm \mathbf{K}_{(f_2,n)-q},$$

$$F_{(f_1 + f_2,m,n)pq} = -\frac{j}{4} \omega_1 \omega_2 (1 + \gamma) u_{(f_1,m)-p} \tilde{u}_{(f_2,n)-q} |\mathbf{K}_{(f_1 + f_2,m,n)pq}|,$$

$$F_{(f_1 - f_2,m,n)pq} = +\frac{j}{4} \omega_1 \omega_2 (1 + \gamma) u_{(f_1,m)-p} \tilde{u}_{(f_2,n)-q} |\mathbf{K}_{(f_1 - f_2,m,n)pq}|. \tag{10}$$

In equation (9), c.c. stands for the complex conjugation of the preceding term, $\mathbf{K}^0_{(f_1 \pm f_2,m,n)pq}$ is an unit vector of $\mathbf{K}_{(f_1 \pm f_2,m,n)pq}$, the subscript, $(f_1 + f_2)$ or $(f_1 - f_2)$, denotes whether the corresponding physical quantity is associated with SF or with DF term, respectively. In this following analysis, for simplicity, we may take into account only the first term of equation (9). i.e., we use the exponential function instead of the cosine function.

Combining equations (7) and (9) yields the driven SF and DF waves

$$\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pq} = \frac{F_{(f_1 \pm f_2,m,n)pq}\mathbf{K}^0_{(f_1 \pm f_2,m,n)pq}}{[(\omega_1 \pm \omega_2)^2 - c^2\mathbf{K}_{(f_1 \pm f_2,m,n)pq}\cdot\mathbf{K}_{(f_1 \pm f_2,m,n)pq}]} \times \exp[\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pq}\cdot\mathbf{r}_{pq} - \mathrm{j}(\omega_1 \pm \omega_2)t],$$

$$\mathbf{K}_{(f_1 \pm f_2,m,n)pq}\cdot\mathbf{r}_{pq} = [k_{(f_1,m)} \pm k_{(f_2,n)}]z + [(-1)^p\alpha_{(f_1,m)}k_{(f_1,m)} \pm (-1)^q\alpha_{(f_2,n)}k_{(f_2,n)}]y, \quad p, q = 1, 2. \quad (11)$$

The wave vectors and the mechanical displacement ones of the driven SF and DF waves are shown in Figure 2. Generally, $(\omega_1 \pm \omega_2)^2 \neq c^2\mathbf{K}_{(f_1 \pm f_2,m,n)pq}\cdot\mathbf{K}_{(f_1 \pm f_2,m,n)pq}$, the amplitude of the driven SF or DF wave is a finite value, i.e., there is no cumulative growth effect for the driven SF and DF waves. Usually, we are interested in the case in which the second order acoustic waves have a cumulative effect. For a driven SF and DF waves, there is a resonant phenomenon once the

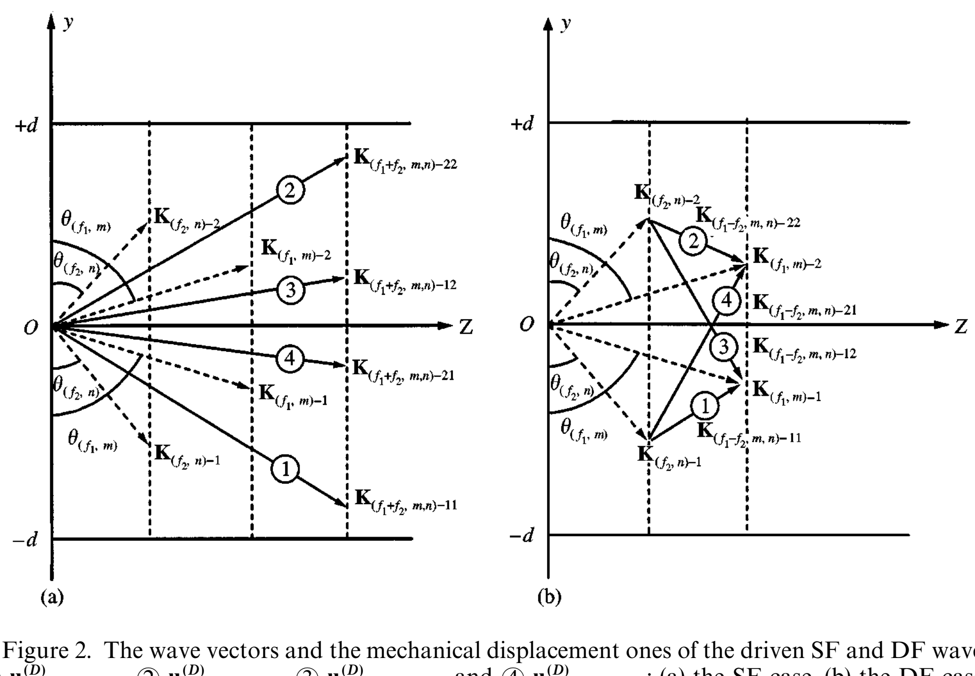

Figure 2. The wave vectors and the mechanical displacement ones of the driven SF and DF waves, ①-$\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)11}$, ②-$\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)22}$, ③-$\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)12}$, and ④-$\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)21}$; (a) the SF case, (b) the DF case.

denominator of the amplitude of $\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pq}$, equals zero. If one take into account the solution to $\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pq}$, the resonance occurs once $(\omega_1 \pm \omega_2)^2 = c^2\mathbf{K}_{(f_1 \pm f_2,m,n)pq} \cdot \mathbf{K}_{(f_1 \pm f_2,m,n)pq}$. It follows that

$$(\omega_1 \pm \omega_2)^2 = c^2|\mathbf{K}_{(f_1,m)-Lp} \pm \mathbf{K}_{(f_2,n)-Lp}|^2, \quad p = 1, 2, \tag{12a}$$

$$(\omega_1 \pm \omega_2)^2 = c^2|\mathbf{K}_{(f_1,m)-Lp} \pm \mathbf{K}_{(f_2,n)-Lq}|^2, \quad p, q = 1, 2,\ p \neq q. \tag{12b}$$

Combining equations (2) and (12) leads to

$$\cos[\theta_{(f_1,m)} - \theta_{(f_2,n)}] = 1, \tag{13a}$$

$$\cos[\theta_{(f_1,m)} + \theta_{(f_2,n)}] = 1. \tag{13b}$$

It is easy from equation (13a) to deduce the relationship $\theta_{(f_1,m)} = \theta_{(f_2,n)}$, i.e., the direction of the wave vector $\mathbf{K}_{(f_1,m)-Lp}$ coincides with that of the wave vector $\mathbf{K}_{(f_2,n)-Lp}$. It is also found, from equation (2), that the phase velocity of the (f_1, m) mode, $c_{(f_1,m)}$, equals that of the (f_2, n) mode, $c_{(f_2,n)}$. Apparently, equation (13b) cannot be satisfied; therefore, there is no resonant phenomenon for the driven SF or DF wave, $\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pq}$ (p, $q = 1, 2$, $p \neq q$). Although the resonance occurs once $\theta_{(f_1,m)} = \theta_{(f_2,n)}$ or $c_{(f_1,m)} = c_{(f_2,n)}$, the solution to $\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pp}$ ($p = 1, 2$) still exists. The solution of the driven SF and DF waves, under the condition $c_{(f_1,m)} = c_{(f_2,n)}$, can be formally given by [4]

$$\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pp} = A\,[\mathbf{K}^0_{(f_1 \pm f_2,m,n)pp} \cdot \mathbf{r}_{pp}]\exp[\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pp} \cdot \mathbf{r}_{pp}], \tag{14}$$

where A is a constant to be determined. After substituting equation (14) into equation (7), we have [the right-hand side of equation (7) is $F_{(f_1 \pm f_2,m,n)pp}\,\mathbf{K}^0_{(f_1 \pm f_2,m,n)pp}$ $\exp(\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pp} \cdot \mathbf{r}_{pp})$]

$$A = \frac{F_{(f_1 \pm f_2,m,n)pp}}{2\mathrm{j}c^2[(\omega_1 \pm \omega_2)/c]}. \tag{15}$$

Combining equations (14), (15), and the relationship $\theta_{(f_1,m)} = \theta_{(f_2,n)}$ or $c_{(f_1,m)} = c_{(f_2,n)}$ yields

$$\begin{aligned}\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pp} &= \frac{F_{(f_1 \pm f_2,m,n)pp}[\mathbf{K}^0_{(f_1 \pm f_2,m,n)pp} \cdot \mathbf{r}_{pp}]}{\mathbf{2}\mathrm{j}c^2[(\omega_1 \pm \omega_2)/c]}\exp[\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pp} \cdot \mathbf{r}_{pp}] \\ &= u^{(D)}_{(f_1 \pm f_2,m,n)pp}[(z/d)\sin\theta_{(f_1,m)} + (-1)^p(y/d)\cos\theta_{(f_2,n)}]\,\mathbf{K}^0_{(f_1,m)-p} \\ &\quad \times \exp[\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pp} \cdot r_{pp}], \\ u^{(D)}_{(f_1 \pm f_2,m,n)pp}, &= \mp\frac{(1+\gamma)\pi^2}{2} \cdot \frac{f_1 d}{c} \cdot \frac{f_2 d}{c} \cdot \frac{u_{(f_1,m)-p}\tilde{u}_{(f_2,n)-p}}{d}, \quad p = 1, 2.\end{aligned} \tag{16}$$

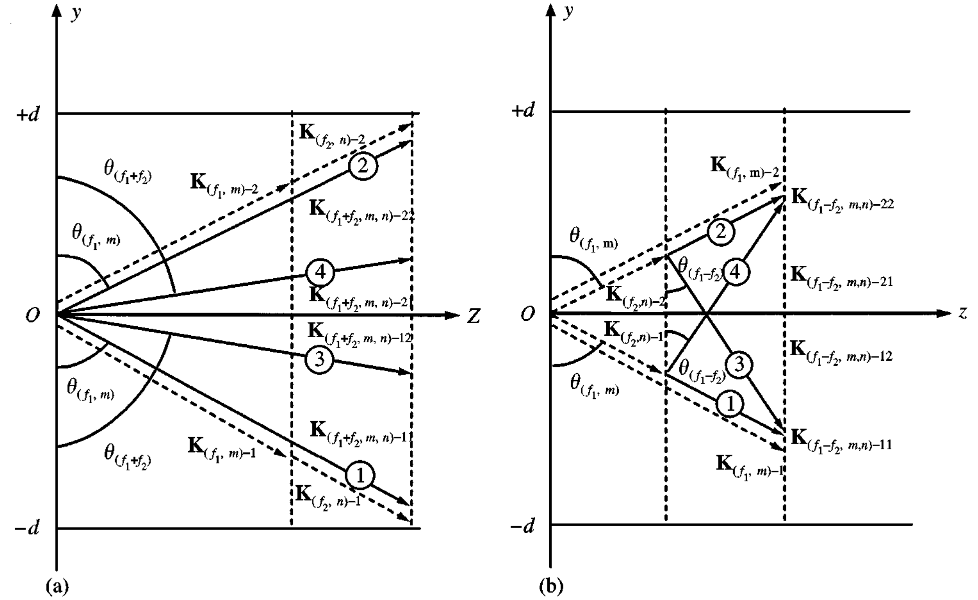

Figure 3. The wave vectors and the mechanical displacement ones of the driven SF and DF waves, as $c_{(f_1,m)} = c_{(f_2,n)}$, ①-$\mathbf{u}^{(D)}_{(f_1\pm f_2,m,n)11}$, ②-$\mathbf{u}^{(D)}_{(f_1\pm f_2,m,n)22}$, ③-$\mathbf{u}^{(D)}_{(f_1\pm f_2,m,n)12}$, and ④-$\mathbf{u}^{(D)}_{(f_1\pm f_2,m,n)21}$; (a) the SF case, (b) the DF case.

Thus the driven SF and DF waves, $\mathbf{u}^{(D)}_{(f_1\pm f_2,m,n)pp}$ $(p = 1, 2)$, grow linearly with the propagation distance once $c_{(f_1,m)} = c_{(f_2,n)}$. For this case there is a relationship

$$\mathbf{K}_{(f_1\pm f_2,m,n)pp} = \mathbf{K}_{(f_1,m)-p} \pm \mathbf{K}_{(f_2,n)-p} = \frac{\omega_1 \pm \omega_2}{c}\mathbf{K}^0_{(f_1,m)-p}. \tag{17}$$

The wave vectors and the corresponding mechanical displacement ones of the driven SF and DF waves with the cumulative effect are shown in Figure 3. There are the driven SF and DF waves without the cumulative growth effect, i.e., $\mathbf{u}^{(D)}_{(f_1\pm f_2,m,n)pq}$ $(p \neq q)$, beside $\mathbf{u}^{(D)}_{(f_1\pm f_2,m,n)pp}$ $(p = 1, 2)$.

Zhou and Shui [7] have revealed that the effect of cumulative growth of reflected second harmonic, at an interface, arises from both the self-interaction of the primary wave (fundamental wave) and the boundary restriction. For the problem of generation of the cumulative SF and DF waves in the waveguide, the partial waves of the (f_1, m) and (f_2, n) modes may be considered to be reflected at the upper and lower walls of the waveguide, i.e., $\mathbf{u}^{(1)}_{(f_1,m)-1}$ and $\mathbf{u}^{(1)}_{(f_2,n)-1}$, may be considered as the reflected waves of $\mathbf{u}^{(1)}_{(f_1,m)-2}$ and $\mathbf{u}^{(1)}_{(f_2,n)-2}$, and *vice versa*. Thus, there are the cumulative SF and DF waves due to the two walls of the waveguide beside $\mathbf{u}^{(D)}_{(f_1\pm f_2,m,n)pp}$ $(p = 1, 2)$.

For the case of the cumulative SF and DF wave generation, there is a boundary condition which requires that the *oy*-axis component of SF and DF mechanical displacement vectors cancel out at the two hard walls. However, this boundary condition cannot be satisfied if we only take into account the driven SF and DF waves. To satisfy this boundary condition, we must introduce the freely propagating SF and DF waves, i.e., the homogeneous solution of equation (7) [8].

There is a condition of phase matching between the driven and the freely propagating SF and DF waves. Thus, the freely propagating SF and DF waves which are to be introduced should propagate along the direction of $\mathbf{K}_{(f_1 \pm f_2,m,n)pp}$ or $\mathbf{K}_{(f_1 \pm f_2,m,n)pq}$. Whereas, there are four driven SF and DF waves under the condition $c_{(f_1,m)} = c_{(f_2,n)}$, and only $\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pp}$ $(p = 1, 2)$ with the wave vector $\mathbf{K}_{(f_1 \pm f_2,m,n)pp} = (\omega_1 \pm \omega_2)\mathbf{K}^0_{(f_1,m)-p}/c$ has a cumulative growth effect (see Figure 3), the freely propagating SF and DF waves propagating along $\mathbf{K}_{(f_1 \pm f_2,m,n)pp}$ should include cumulative terms along the oy- and oz-axes. However, the freely propagating SF and DF waves with the wave vector $\mathbf{K}_{(f_1 \pm f_2,m,n)pq}$ do not include cumulative terms because $\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pq}$ is independent of propagation distance. Considering the oz-axis phase matching between the driven and the freely propagating SF and DF waves, we describe the freely propagating SF and DF waves propagating along $\mathbf{K}_{(f_1 \pm f_2,m,n)pp}$ $(p = 1, 2)$ and $\mathbf{K}_{(f_1 \pm f_2,m,n)pq}$ $(p, q = 1, 2, p \neq q)$ as follows [4, 7]:

$$\mathbf{u}^{(F)}_{(f_1 \pm f_2,m,n)pq} = \sum_{p=1}^{2} \mathbf{u}^{(FC)}_{(f_1 \pm f_2,m,n)pp} + \sum_{p,q=1}^{2} \mathbf{u}^{(FP)}_{(f_1 \pm f_2,m,n)pq},$$

$$\mathbf{u}^{(FC)}_{(f_1 \pm f_2,m,n)pp} = u^{(FC)}_{(f_1 \pm f_2,m,n)pp}\left[(z/d)\cos\theta_{(f_1,m)} + (-1)^{p-1}(y/d)\sin\theta_{(f_1,m)}\right] \times \mathbf{K}^0_{(f_1 \pm f_2,m,n)pp} \exp\left[\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pp} \cdot \mathbf{r}_{pp}\right],$$

$$\mathbf{u}^{(FP)}_{(f_1 \pm f_2,m,n)pq} = u^{(FP)}_{(f_1 \pm f_2,m,n)pq}\mathbf{K}^0_{(f_1 \pm f_2,m,n)pq} \exp\left[\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pq} \cdot \mathbf{r}_{pq}\right], \tag{18}$$

where $u^{(FC)}_{(f_1 \pm f_2,m,n)pp}$ is the amplitude of the freely propagating cumulative wave, and $u^{(FP)}_{(f_1 \pm f_2,m,n)pq}$ the amplitude of the freely propagating plane wave.

The ultimate SF and DF waves consist of the driven and the freely propagating SF and DF waves. It follows that

$$\mathbf{u}^{(2)}_{(f_1 \pm f_2,m,n)} = \sum_{p=1}^{2} \mathbf{u}^{(2)}_{(f_1 \pm f_2,m,n)pp} + \sum_{p \neq q, p,q=1,2} \mathbf{u}^{(2)}_{(f_1 \pm f_2,m,n)pq} \tag{19}$$

with

$$\mathbf{u}^{(2)}_{(f_1 \pm f_2,m,n)pp} = \Big\{u^{(D)}_{(f_1 \pm f_2,m,n)pp}\left[(z/d)\sin\theta_{(f_1,m)} + (-1)^{p}(y/d)\cos\theta_{(f_1,m)}\right] + u^{(FC)}_{(f_1 \pm f_2,m,n)pp}\left[(z/d)\cos\theta_{(f_1,m)} + (-1)^{p-1}(y/d)\sin\theta_{(f_1,m)}\right] + u^{(FP)}_{(f_1 \pm f_2,m,n)pp}\Big\}\,\mathbf{K}^0_{(f_1,m)-p} \exp\left[\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pp} \cdot \mathbf{r}_{pp}\right],$$

$$\mathbf{u}^{(2)}_{(f_1 \pm f_2,m,n)pq} = \Big\{u^{(D)}_{(f_1 \pm f_2,m,n)pq} + u^{(FP)}_{(f_1 \pm f_2,m,n)pq}\Big\}\mathbf{K}^{(0)}_{(f_1 \pm f_2,m,n)pq} \exp\left[\mathrm{j}\mathbf{K}_{(f_1 \pm f_2,m,n)pq} \cdot \mathbf{r}_{pq}\right], \tag{20}$$

where $u^{(D)}_{(f_1 \pm f_2,m,n)pq}$ is the amplitude of $\mathbf{u}^{(D)}_{(f_1 \pm f_2,m,n)pq}$ shown in equation (16). The SF and DF wave boundary condition requires that the oy-axis component of $\mathbf{u}^{(2)}_{(f_1 \pm f_2,m,n)}$ equal zero. Thus, we have

$$[M(\alpha_{(f_1,m)}k_{(f_1,m)}d \pm \alpha_{(f_2,n)}k_{(f_2,n)}d)]\begin{bmatrix} u^{(D)}_{(f_1 \pm f_2,m,n)11}\sin\theta_{(f_1,m)} + u^{(FC)}_{(f_1 \pm f_2,m,n)11}\cos\theta_{(f_1,m)} \\ u^{(D)}_{(f_1 \pm f_2,m,n)22}\sin\theta_{(f_1,m)} + u^{(FC)}_{(f_1 \pm f_2,m,n)22}\cos\theta_{(f_1,m)} \end{bmatrix}$$

$$\times\left(\frac{z}{d}\right) + [M(\alpha_{(f_1,m)}k_{(f_1,m)}d \pm \alpha_{(f_2,n)}k_{(f_2,n)}d)]\begin{bmatrix} u^{(FP)}_{(f_1 \pm f_2,m,n)11} \\ u^{(FP)}_{(f_1 \pm f_2,m,n)22} \end{bmatrix}$$

$$+ [M(\alpha_{(f_1,m)}k_{(f_1,m)}d \mp \alpha_{(f_2,n)}k_{(f_2,n)}d)]\begin{bmatrix} u^{(FP)}_{(f_1 \pm f_2,m,n)12} + u^{(D)}_{(f_1 \pm f_2,m,n)12} \\ u^{(FP)}_{(f_1 \pm f_2,m,n)21} + u^{(D)}_{(f_1 \pm f_2,m,n)21} \end{bmatrix}$$

$$\times\frac{\sin\theta_{(f_1 \pm f_2)}}{\sin\theta_{(f_1,m)}} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}, \tag{21}$$

where the right-hand side of equation (21) is presented in Appendix A, and $\theta_{(f_1 \pm f_2)}$ is shown in Figure 3. Equation (21) should be satisfied at any point on the walls of the waveguide, which means that [4, 7]

$$[M(\alpha_{(f_1,m)}k_{(f_1,m)}d \pm \alpha_{(f_2,n)}k_{(f_2,n)}d)]$$

$$\times\begin{bmatrix} u^{(D)}_{(f_1 \pm f_2,m,n)11}\sin\theta_{(f_1,m)} + u^{(FC)}_{(f_1 \pm f_2,m,n)11}\cos\theta_{(f_1,m)} \\ u^{(D)}_{(f_1 \pm f_2,m,n)22}\sin\theta_{(f_1,m)} + u^{(FC)}_{(f_1 \pm f_2,m,n)22}\cos\theta_{(f_1,m)} \end{bmatrix} = 0 \tag{22}$$

and

$$[M(\alpha_{(f_1,m)}k_{(f_1,m)}d \pm \alpha_{(f_2,n)}k_{(f_2,n)}d)]\begin{bmatrix} u^{(FP)}_{(f_1 \pm f_2,m,n)11} \\ u^{(FP)}_{(f_1 \pm f_2,m,n)22} \end{bmatrix}$$

$$+ [M(\alpha_{(f_1,m)}k_{(f_1,m)}d \mp \alpha_{(f_2,n)}k_{(f_2,n)}d)]\begin{bmatrix} u^{(FP)}_{(f_1 \pm f_2,m,n)12} + u^{(D)}_{(f_1 \pm f_2,m,n)12} \\ u^{(FP)}_{(f_1 \pm f_2,m,n)21} + u^{(D)}_{(f_1 \pm f_2,m,n)21} \end{bmatrix}$$

$$\times\frac{\sin\theta_{(f_1 \pm f_2)}}{\sin_{(f_1,m)}} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}. \tag{23}$$

Taking into account the form of the coefficients matrices of equations (22) and (23), and the dispersion relationships, i.e., $2\alpha_{(f_1,m)}k_{(f_1,m)}d = m\pi$ and $2\alpha_{(f_2,n)}k_{(f_2,n)}d = n\pi$, we have

$$|M(\alpha_{(f_1,m)}k_{(f_1,m)}d \pm \alpha_{(f_2,n)}k_{(f_2,n)}d)| = 0,$$

$$|M(\alpha_{(f_1,m)}k_{(f_1,m)}d \mp \alpha_{(f_2,n)}k_{(f_2,n)}d)| = 0. \tag{24}$$

Equation (22) has a non-trivial solution. In this article, interest is focused on the physical process of generation of the cumulative SF and DF waves. Combining equations (5), (22), and (24) yields

$$[u^{(D)}_{(f_1 \pm f_2,m,n)11} \sin\theta_{(f_1,m)} + u^{(FC)}_{(f_1 \pm f_2,m,n)11} \cos\theta_{(f_1,m)}]$$

$$= (-1)^{m+n}[u^{(D)}_{(f_1 \pm f_2,m,n)22} \sin\theta_{(f_1,m)} + u^{(FC)}_{(f_1 \pm f_2,m,n)22} \cos\theta_{(f_1,m)}]. \quad (25)$$

From equations (6) and (16) we have $u^{(D)}_{(f_1 \pm f_2,m,n)11} = (-1)^{(m+n)} u^{(D)}_{(f_1 \pm f_2,m,n)22}$, and then, the relationship $u^{(FC)}_{(f_1 \pm f_2,m,n)11} = (-1)^{(m+n)} u^{(FC)}_{(f_1 \pm f_2,m,n)22}$. Thus, we can draw a conclusion that the ultimate cumulative SF and DF waves (including the driven and the freely propagating waves) are symmetrical as $m + n =$ even number, and antisymmetrical as $m + n =$ odd number, i.e., the characteristics of symmetry of the cumulative SF and DF waves are determined by those of the (f_1, m) and (f_2, n) modes.

Next we derive the analytical solution of the cumulative SF and DF waves in the waveguide. The cumulative SF and DF wave component along the oz-axis, denoted by $u^{(2C)}_{(f_1 \pm f_2,m,n)z}$, is given by [neglecting the factor $\exp[j(k_{(f_1,m)} \pm k_{(f_2,n)})z]$]

$$u^{(2C)}_{(f_1 \pm f_2,m,n)z} = \sin\theta_{(f_1,m)} \left\{ \begin{array}{l} [\sin\theta_{(f_1,m)} u^{(D)}_{(f_1 \pm f_2,m,n)11} + \cos\theta_{(f_1,m)} u^{(FC)}_{(f_1 \pm f_2,m,n)11}] \dfrac{z}{d} \\ - [\cos\theta_{(f_1,m)} u^{(D)}_{(f_1 \pm f_2,m,n)11} - \sin\theta_{(f_1,m)} u^{(FC)}_{(f_1 \pm f_2,m,n)11}] \dfrac{y}{d} \end{array} \right\} R_-$$

$$+ \sin\theta_{(f_1,m)} \left\{ \begin{array}{l} [\sin\theta_{(f_1,m)} u^{(D)}_{(f_1 \pm f_2,m,n)22} + \cos\theta_{(f_1,m)} u^{(FC)}_{(f_1 \pm f_2,m,n)22}] \dfrac{z}{d} \\ + [\cos\theta_{(f_1,m)} u^{(D)}_{(f_1 \pm f_2,m,n)22} - \sin\theta_{(f_1,m)} u^{(FC)}_{(f_1 \pm f_2,m,n)22}] \dfrac{y}{d} \end{array} \right\} R_+, \quad (26)$$

where R_+ and R_- are presented in Appendix A.

We assume that the (f_1, m) and (f_2, n) modes are radiated by a pistonlike excitation source at $z = 0$. The initial condition of excitation requires that $u^{(2C)}_{(f_1 \pm f_2,m,n)z} = 0$ at $z = 0$. Thus equation (26) leads to [3, 4, 7, 9]

$$u^{(FC)}_{(f_1 \pm f_2,m,n)pp} = \frac{\cos\theta_{(f_1,m)}}{\sin\theta_{(f_1,m)}} u^{(D)}_{(f_1 \pm f_2,m,n)pp}, \quad p = 1, 2. \quad (27)$$

Now we have determined the ultimate cumulative SF and DF waves by equations (16), (20) and (27). The cumulative SF and DF waves associated with the (f_1, m) and

(f_2, n) modes can be written as

$$\mathbf{u}^{(2C)}_{(f_1 \pm f_2, m,n)} = \mathbf{u}^{(2C)}_{(f_1 \pm f_2, m,n)11} + \mathbf{u}^{(2C)}_{(f_1 \pm f_2, m,n)22},$$

$$\mathbf{u}^{(2C)}_{(f_1 \pm f_2, m,n)pp} = u^{(D)}_{(f_1 \pm f_2, m,n)pp} \frac{z}{d \sin\theta_{(f_1, m)}} \times \mathbf{K}^0_{(f_1, m)-p} \exp[\mathrm{j}(\omega_1 \pm \omega_2)\mathbf{K}^0_{(f_1, m)-p} \cdot \mathbf{r}_p/c], \quad p = 1, 2. \tag{28}$$

Equation (28) satisfying the boundary and initial conditions of excitation is the solution to be determined.

From equation (27) it is easy to show that the right-hand side of equation (23) equals zero. On the basis of equation (24), equation (23) yields two matrix equations,

$$[M(\alpha_{(f_1, m)} k_{(f_1, m)} d \pm \alpha_{(f_2, n)} k_{(f_2, n)} d)] \begin{bmatrix} u^{(FP)}_{(f_1 \pm f_2, m,n)11} \\ u^{(FP)}_{(f_1 \pm f_2, m,n)22} \end{bmatrix} = 0,$$

$$[M(\alpha_{(f_1, m)} k_{(f_1, m)} d \mp \alpha_{(f_2, n)} k_{(f_2, n)} d)] \begin{bmatrix} u^{(FP)}_{(f_1 \pm f_2, m,n)12} + u^{(D)}_{(f_1 \pm f_2, m,n)12} \\ u^{(FP)}_{(f_1 \pm f_2, m,n)21} + u^{(D)}_{(f_1 \pm f_2, m,n)21} \end{bmatrix} = 0. \tag{29}$$

Equations (24) and (29) show that the freely propagating plane waves have a non-trivial solution. However, there is a shortcoming that the freely propagating plane waves cannot be fully found in the present analysis. The cumulative SF or DF wave plays a dominant role due to its cumulative growth effect. It is less important that the freely propagating plane waves cannot be determined.

In practical cases, the solution should be the real part of $\mathbf{u}^{(2C)}_{(f_1 \pm f_2, m,n)}$ shown in equation (28), i.e., $(\mathbf{u}^{(2C)}_{(f_1 \pm f_2, m,n)} + \tilde{\mathbf{u}}^{(2C)}_{(f_1 \pm f_2, m,n)})/2$, where $\tilde{\mathbf{u}}^{(2C)}_{(f_1 \pm f_2, m,n)}$ is the complex conjugation of $\mathbf{u}^{(2C)}_{(f_1 \pm f_2, m,n)}$.

3. QUANTITATIVE ANALYSIS

On the basis of equation (5) we have the following dispersion equation:

$$\frac{c_{(f_i, l)}}{c} = \frac{4 f_i d/c}{\sqrt{16 (f_i d/c)^2 - l^2}}, \quad (l = 0, 1, 2, 3, \ldots). \tag{30}$$

Figure 4 shows the dispersion curves corresponding to equation (30). The beeline L is parallel to the horizontal axis. There are a series of cross points between the beeline L and the dispersion curves. For example, we take into account the two cross-points A and B with the co-ordinates $A(f_1 d, c_{(f_1, m)}, m) = (0{\cdot}95c, 1{\cdot}64c, 3)$ and $B(f_2 d, c_{(f_2, n)}, n) = (0{\cdot}63c, 1{\cdot}64c, 2)$. From equation (2) the angles $\theta_{(f_1, m)}$ and $\theta_{(f_2, n)}$ can be calculated. We assume an excitation source with two frequencies f_1 and f_2 is

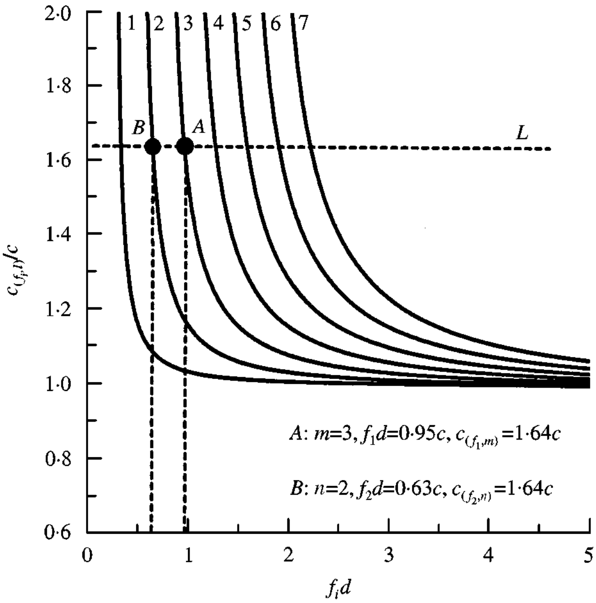

Figure 4. The dispersion curves of the fundamental modes.

piston-like at $z = 0$, and only radiates two fundamental modes, i.e., the (f_1, m) and (f_2, n) modes. Because of $c_{(f_1,m)} = c_{(f_2,n)}$, the SF and DF waves in the waveguide have the cumulative growth effect.

Next we will use a numerical computation to illustrate the fields of the cumulative SF and DF waves. For simplicity, the fluid in the waveguide is assumed to be an ideal gas with the constant values $c = 334$ m/s and $\gamma = 1{\cdot}4$. On the basis of equation (28), the cumulative SF and DF wave displacement components along the oy- and oz-axis, denoted by $u_y^{(2C)}$ and $u_z^{(2C)}$, are given by

$$u_y^{(2C)} = \sum_{p=1}^{2} (-1)^p \frac{\cos\theta_{(f_1,m)}}{\sin\theta_{(f_1,m)}} \frac{z}{d} u^{(D)}_{(f_1 \pm f_2,m,n)pp} \exp\left[\mathrm{j}(\omega_1 \pm \omega_2)\mathbf{K}^0_{(f_1,m)-p} \cdot \mathbf{r}_p/c\right] + \text{c.c.},$$

$$u_z^{(2C)} = \sum_{p=1}^{2} \frac{z}{d} u^{(D)}_{(f_1 \pm f_2,m,n)pp} \exp\left[\mathrm{j}(\omega_1 \pm \omega_2)\mathbf{K}^0_{(f_1,m)-p} \cdot \mathbf{r}_p/c\right] + \text{c.c.} \tag{31}$$

In equation (31), "c.c." is the complex conjugation of the preceding term, $u^{(D)}_{(f_1 \pm f_2,m,n)pp}$ $(p = 1, 2)$ is determined by equation (16), the angle $\theta_{(f_1,m)}$ can be calculated by equation (2) if $f_i d$ $(i = 1, 2)$, m (or n), and $c_{(f_1,m)}$ are given. At the moment, it is easy, from equation (31), to calculate the values of $u_y^{(2C)}$ and $u_z^{(2C)}$ versus (x, y). Figure 5 shows the corresponding field patterns of the cumulative SF and DF waves [corresponding to $(u_{(f_1,m)-1}\tilde{u}_{(f_2,n)-1} + \tilde{u}_{(f_1,m)-1}u_{(f_2,n)-1})/d$]. It is easy to see that the cumulative SF and DF wave field patterns possess the effect of

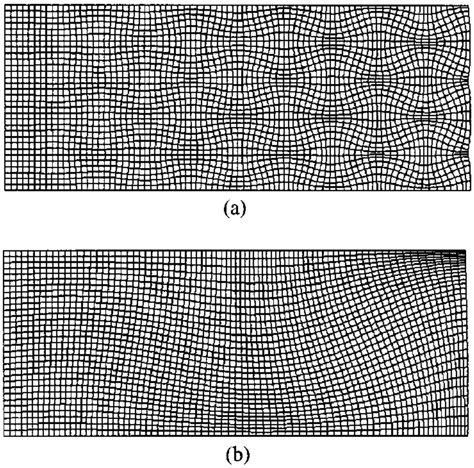

Figure 5. The cumulative SF and DF wave field patterns, $f_1 d = 0{\cdot}95c$, $f_2 d = 0{\cdot}63c$, $c_{(f_1,m)} = c_{(f_2,n)} = 1{\cdot}64c$, $m = 3$, $n = 2$; (a) the SF case, (b) the DF case.

two-dimensional cumulative growth, and that the corresponding field patterns are antisymmetrical (due to $m = 3$, $n = 2$).

4. CONCLUSION

As shown above, we originate a comprehensive theory to analyze the generation of the cumulative SF and DF waves in a two-dimensional hard-walled waveguide. If an excitation source is of multi-frequency, there will be the SF and DF wave generation due to the bulk non-linearity of the fluid in the waveguide. In this article, for simplicity, we assume an excitation source includes two frequencies f_1 and f_2, and generates two sets of the fundamental modes, i.e., the (f_1, m) and (f_2, n) modes. The cross-interaction of the partial waves of the (f_1, m) and (f_2, n) modes generates the SF and DF waves in the waveguide. The results show that the driven SF and DF waves retain the cumulative effect once the two fundamental modes with different frequencies have the same phase velocities. On the basis of the analysis method of non-linear acoustic waves at an interface, and the boundary and initial conditions of excitation, the analytical expression of the cumulative SF and DF waves has been obtained. It is also found that the symmetrical characteristics of the cumulative SF and DF wave fields are determined by those of the two fundamental modes. Moreover, we illustrate the procedure to calculate the ultimate SF and DF wave field patterns in the waveguide if a pistonlike excitation source at an initial plane is known.

As for the discussion on the generation of the cumulative SF and DF waves in the waveguide, the physical model and the analysis process are clear. From the viewpoint of second order perturbation, we can solve the problem of the cumulative SF and DF wave generation accurately, if second order approximation is adequate.

The analysis process described in this article lays a foundation for studying the cumulative wave generation of time-domain acoustic fields that may be of multi-frequency.

REFERENCES

1. A. H. NAYFEN and MING-SHING TSAI 1974 *Journal of the Acoustical Society of America* **55**, 1166–1172. Nonlinear acoustic propagation in two-dimensional ducts.
2. J. H. GINSBERG 1979 *Journal of the Acoustical Society of America* **65**, 1127–1133. Finite-amplitude two-dimensional waves in a rectangular duct induced by arbitrary periodic excitation.
3. M. X. DENG 1998 *Journal of Sound and Vibration* **218**, 791–798. Propagation of finite amplitude acoustic waves in a two-dimensional rectangular duct.
4. M. DENG 1998 *Journal of Applied Physics* **83**, 7461–7466. Generation of second-order cumulative waves accompanying the fundamental modes in a two-dimensional hard-walled waveguide.
5. B. A. AULD 1973 *Acoustic fields and Waves in Solids*, Vol. 2. New York: Wiley.
6. O. V. RUDENKO and S. I. SOLUYAN 1977 *Theoretical Foundation of Nonlinear Acoustics*. New York: Consultants Bureau.
7. S. ZHOU and Y. SHUI 1992 *Journal of Applied Physics* **72**, 5070–5080. Nonlinear reflection of bulk acoustic waves at an interface.
8. Y. SHUI, I. YU, SOLODOV 1985 *Proceedings of the second West Pacific Regional Acoustics Conference*, Vol. B1, 188–192. Nonlinear reflection of bulk acoustic waves in solids.
9. Y. SHUI, I. YU. SOLODOV 1988 *Journal of Applied Physics* **64**, 6155–6165. Nonlinear properties of Rayleigh and Stoneley waves in solids.

APPENDIX A

The coefficient matrices $[M(k_{(f_i,l)}d)]$, $[M(\alpha_{(f_1,m)}k_{(f_1,m)}d \pm \alpha_{(f_2,n)}k_{(f_2,n)}d)]$, and $[M(\alpha_{(f_1,m)}k_{(f_1,m)}d \mp \alpha_{(f_2,n)}k_{(f_2,n)}d)]$ are, respectively, given by

$$[M(k_{(f_i,l)}d)] = \begin{bmatrix} -\exp(-j\alpha_{(f_i,l)}k_{(f_i,l)}d) & +\exp(+j\alpha_{(f_i,l)}k_{(f_i,l)}d) \\ -\exp(+j\alpha_{(f_i,l)}k_{(f_i,l)}d) & +\exp(-j\alpha_{(f_i,l)}k_{(f_i,l)}d) \end{bmatrix},$$

$$[M(\alpha_{(f_1,m)}k_{(f_1,m)}d \pm \alpha_{(f_2,n)}k_{(f_2,n)}d)] = \begin{bmatrix} -R_- & R_+ \\ -R_+ & R_- \end{bmatrix}$$

and

$$[M(\alpha_{(f_1,m)}k_{(f_1,m)}d \mp \alpha_{(f_2,n)}k_{(f_2,n)}d)] = \begin{bmatrix} -R'_- & R'_+ \\ -R'_+ & R'_- \end{bmatrix}.$$

where $R_+ = \exp[+j(\alpha_{(f_1,m)}k_{(f_1,m)}d \pm \alpha_{(f_2,n)}k_{(f_2,n)}d)]$, $R_- = \exp[-j(\alpha_{(f_1,m)}k_{(f_1,m)}d \pm \alpha_{(f_2,n)}k_{(f_2,n)}d)]$, $R'_+ = \exp[+j(\alpha_{(f_1,m)}k_{(f_1,m)}d \mp \alpha_{(f_2,n)}k_{(f_2,n)}d)]$, and $R'_- = \exp[-j(\alpha_{(f_1,m)}k_{(f_1,m)}d \mp \alpha_{(f_2,n)}k_{(f_2,n)}d)]$,

The right-hand side of equation (21) is expressed as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} +R_- & +R_+ \\ -R_+ & -R_- \end{bmatrix} \begin{bmatrix} \cos\theta_{(f_1,m)}\, u^{(D)}_{(f_1 \pm f_2,m,n)11} - \sin\theta_{(f_1,m)} u^{(FC)}_{(f_1 \pm f_2,m,n)11} \\ \cos\theta_{(f_1,m)}\, u^{(D)}_{(f_1 \pm f_2,m,n)22} - \sin\theta_{(f_1,m)} u^{(FC)}_{(f_1 \pm f_2,m,n)22} \end{bmatrix}.$$